Numerical Aspects of the Implementation of Artificial Boundary Conditions for Laminar Fluid-Structure Interactions

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FACULTÉ DES SCIENCES
Département de physique théorique

Excerpt from the Proceedings of the 2012 COMSOL Conference in Milan

Modelling

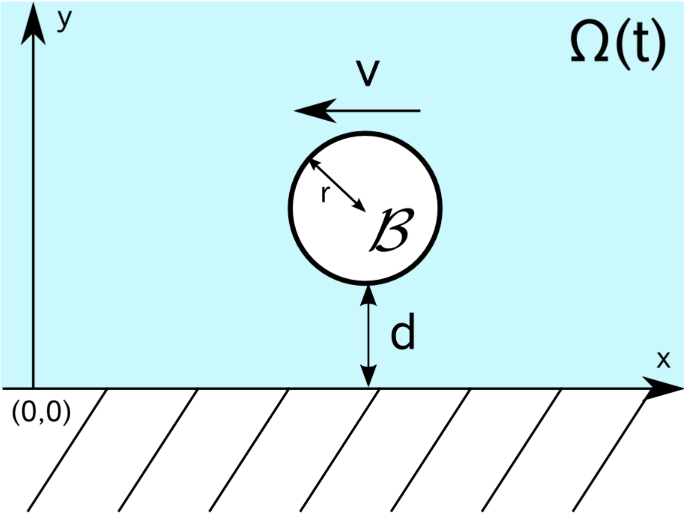

Figure: Moving bubble

Modelling

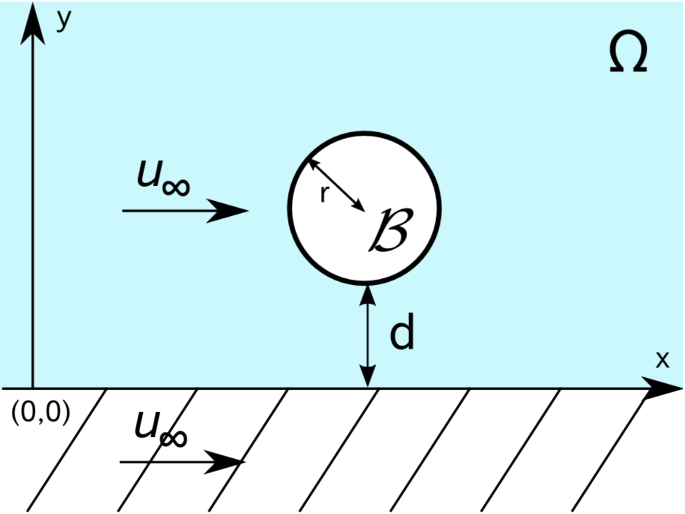

Figure: Moving fluid & wall

Mathematical description

The steady Navier-Stokes equations in the halfspace

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \mathrm{Re}^{-1} \Delta \boldsymbol{u} = 0 \ ,$$
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in $\Omega = \mathbb{R}^2_{y \geq 0} \setminus \mathcal{B}$, with

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Clean and contaminated bubbles

Boundary conditions at interface

- clean: slip b.c.

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Code & hardware

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- 12-core PC (36 MB RAM)

Truncating the domain

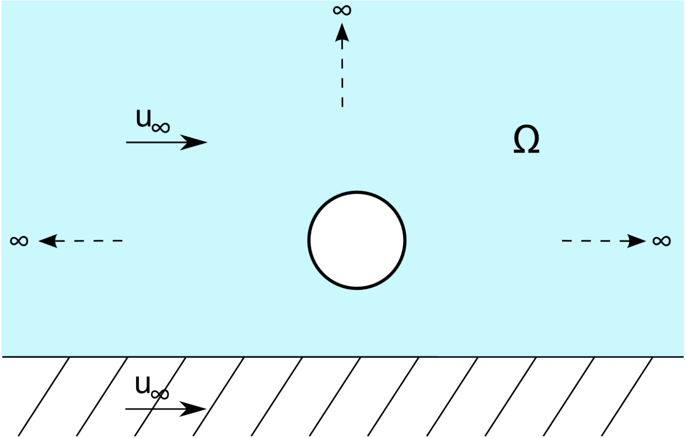

Figure: Theoretical domain.

Truncating the domain

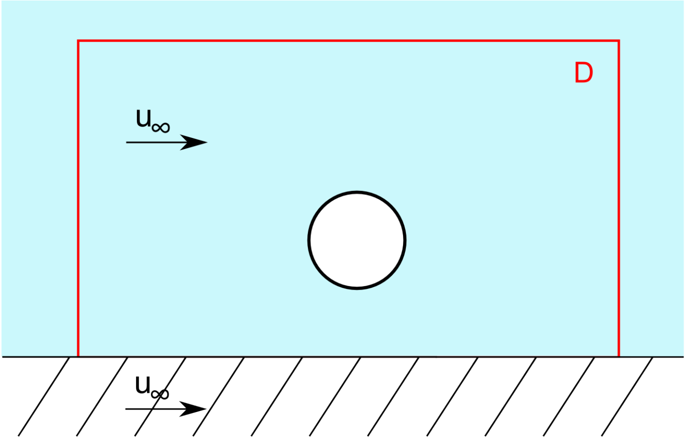

Figure: Numerical domain.

Truncating the domain

Problems and consequences

- What boundary conditions on the artificial boundaries?

Truncating the domain

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- What boundary conditions on the artificial boundaries?
- What impact on physical quantities like drag and lift?

The domain

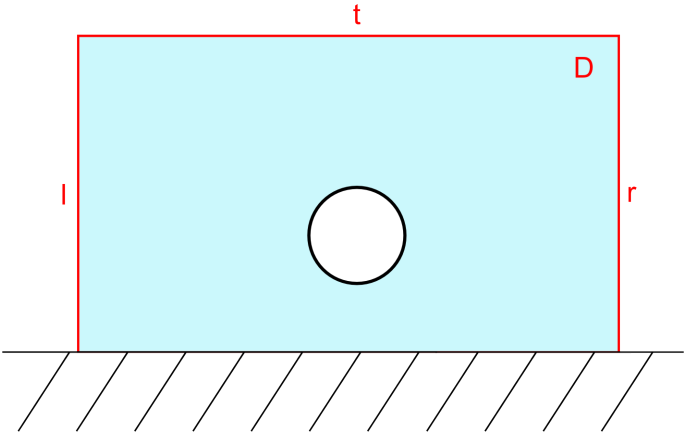

Figure: The numerical domain and its boundaries.

Simple boundary conditions

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Fixed inflow: Overestimation of flow rate!

Classic boundary conditions

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Normal stress ($f_0 = 0$) at the "open" boundaries
$\boldsymbol{u}_\infty$ at "inlet"

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Is the hypothesis sound at all?

Adaptive boundary conditions

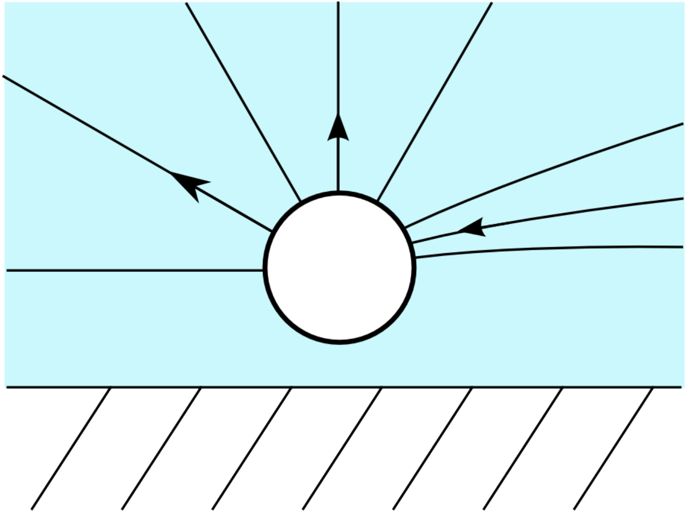

Figure: Potential and wake component (streamlines).

Adaptive boundary conditions

Asymptotic series expansion

$$\frac{u_{\mathrm{as}}}{u_\infty} = 1$$

$$\frac{v_{\mathrm{as}}}{u_\infty} = 0$$

Adaptive boundary conditions

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$$\frac{u_{\mathrm{as}}}{u_\infty} = 1 + c_1 \frac{1}{y^{\frac{3}{2}}} \varphi\left(\frac{x}{y}\right)$$

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Adaptive boundary conditions

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$$\frac{u_{\mathrm{as}}}{u_\infty} = 1 + c_1 \frac{1}{y^{\frac{3}{2}}} \varphi\left(\frac{x}{y}\right) + c_1 \frac{1}{y^2} \eta\left(\frac{x}{y^2}\right) + \dots$$

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Explicit functions (see conference proceedings)!

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- $\Rightarrow$ run-time iterative determination of c_1 (algorithm)

Streamlines of $\boldsymbol{u} - \boldsymbol{u}_\infty$

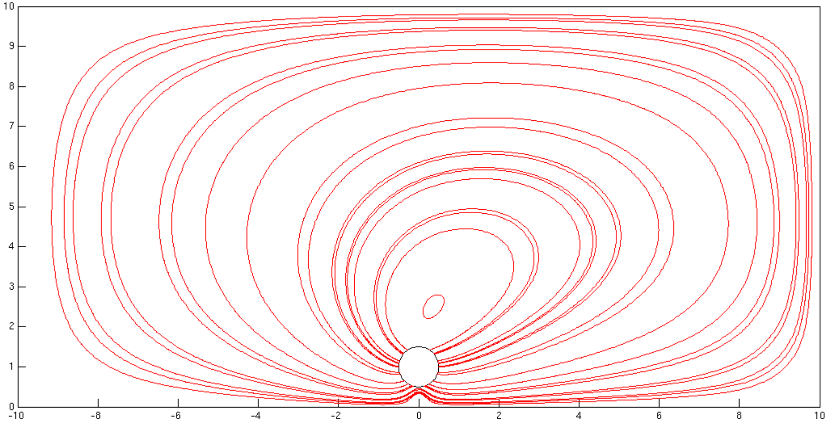

Figure: Simple boundary conditions, D_{10}.

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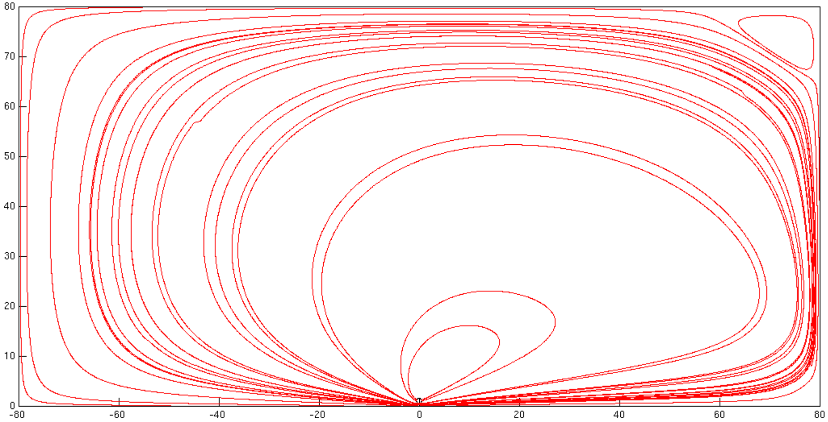

Figure: Simple boundary conditions, D_{80}.

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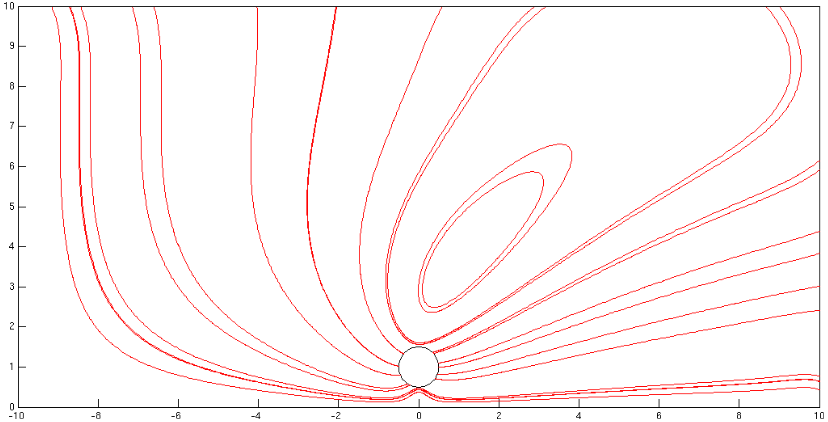

Figure: Classic boundary conditions, D_{10}.

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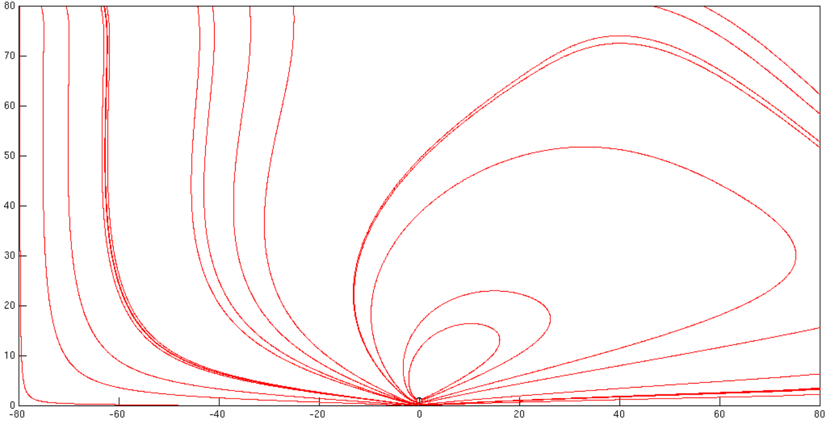

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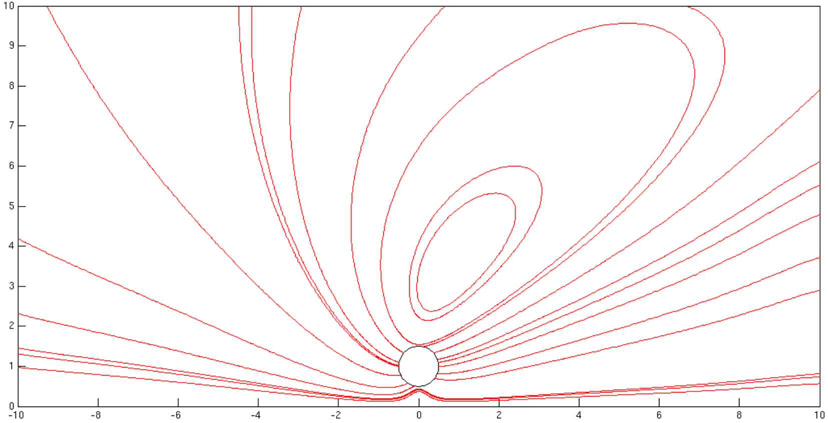
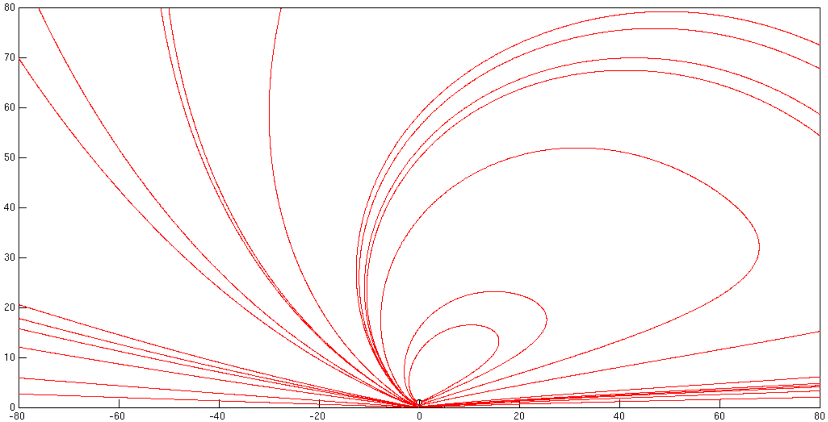


Figure: Adaptive boundary conditions, D_{10}.

Streamlines of $\boldsymbol{u} - \boldsymbol{u}_\infty$



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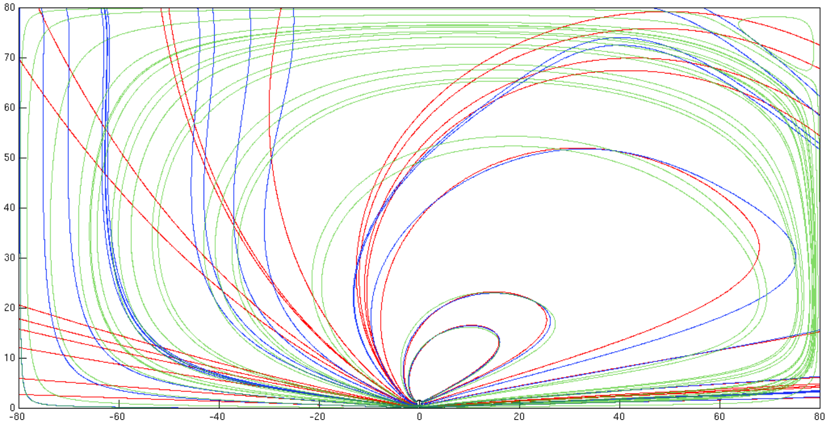


Figure: $\boldsymbol{u} - \boldsymbol{u}_\infty$, overlay of the three b.c. cases, D_{80}.

The case of a clean bubble

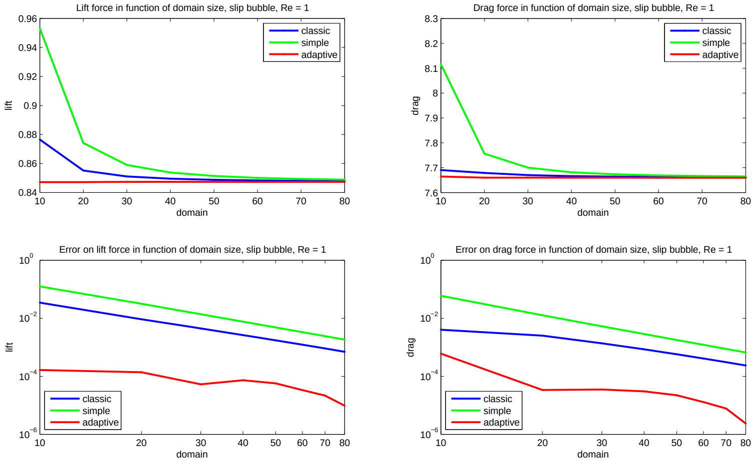
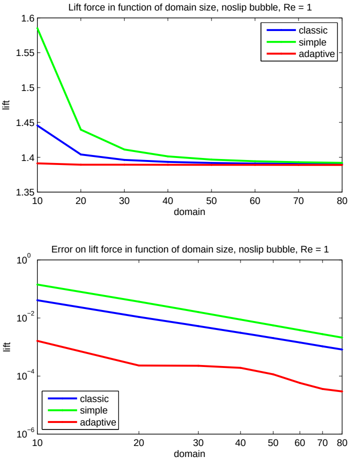
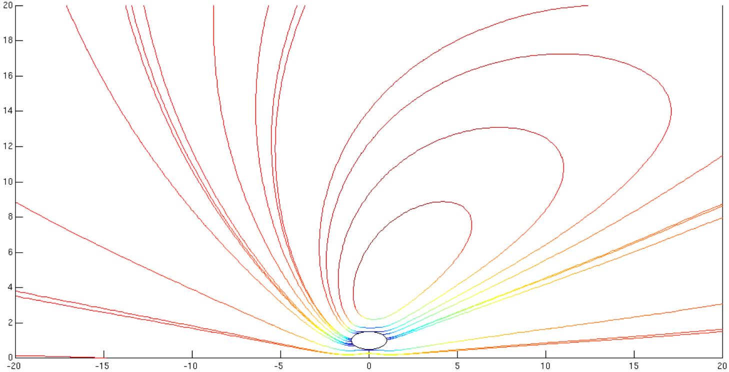


Figure: Reference: D_{90} & a.b.c.

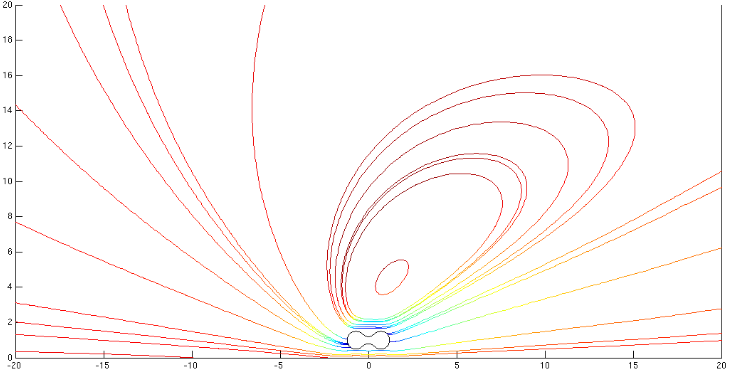
The case of a contaminated bubble



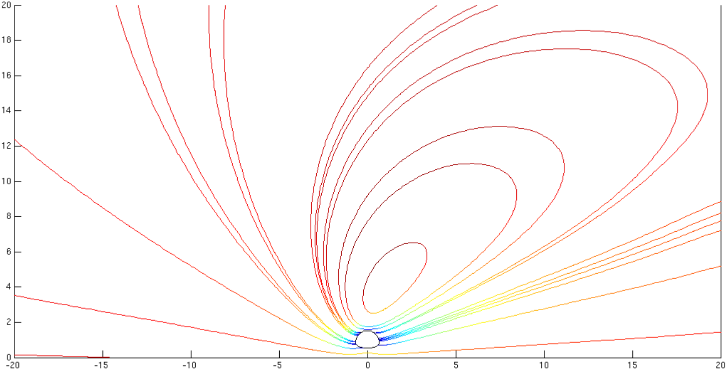
Ellipse



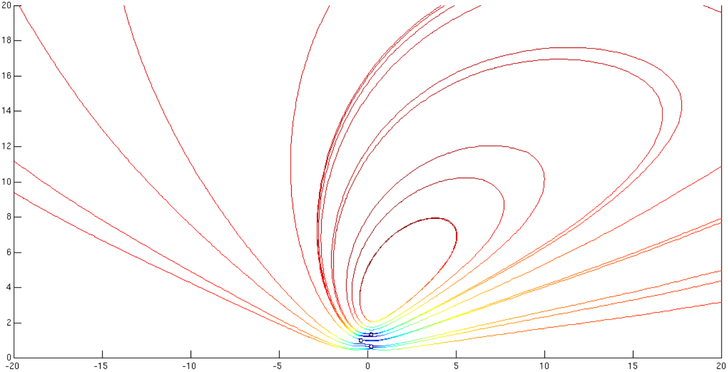
"Bone"



Asymmetric



Multiple disks



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References

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- CFD: C.B. & Wittwer, submitted to *C&F* (2012)
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