Structure Of A Classical Vortex Ring

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Abstract:

As in Ref. 1 the present COMSOL model simulates an axisymmetric motion witout swirl of an incompressible inviscid fluid of uniform density, namely a vortex ring whose motion is steady in time when viewed by a suitably propagating observer. The model assumes that the domain is unbounded, the motion is irrotational exterior to the core boundary, and the azimuthal component of the curl of the velocity is directly proportional to the distance from the axis of symmetry. By allowing a typical meridional section of the core to be noncircular the new results satisfy the condition of no-slip across the core boundary.

Keywords: Unbounded domains, Vortex ring, Kelvin Inversion, Optimizaton, Boundary PDE

1. Three domains and the field equations that hold within them

Let $\mathcal{R}$ be an unbounded region consisting of all of physical space and suppose that $\mathcal{R}$ is filled with a uniform-density incompressible, inviscid fluid. Assume that the motion is axisymmetric, that the motion due to a vortex ring, and that the fluid is at rest at sufficiently large distances from that ring. Let $\mathbf{u}$ denote the fluid velocity relative to the remote undisturbed fluid. Then the assumption of incompressibility implies that $\mathbf{u}$ is solenoidal (*i.e.* $\operatorname{div}\mathbf{u} = 0$) everywhere. Assume that the subregion $\mathcal{R}^c$ in which the motion is rotational (*i.e.* $\operatorname{curl}\mathbf{u} \neq \mathbf{0}$) is a bounded *vortex core*.

Let (x, y, z) be cartesian coordinates whose z-axis coincides with the axis of symmetry. I assume that $\mathcal{R}^c$ has a plane of symmetry—the *equitorial plane*—perpendicular to the z-axis and that the origin of the axis system is situated on the equitorial plane. Let (r, ϕ, z) be cylindrical coordinates related to the cartesian coordinates in the usual way, , *i.e.* $(x, y) = (r\cos\phi, r\sin\phi)$ and let $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{k}}\}$ be the right-handed system of unit vectors in the directions of increasing r, ϕ, and z, respectively. Let (R, θ, ϕ) be spherical coordinates related to the cylindrical coordinates in the usual way, *i.e.* $(z, r) = (R\cos\theta, R\sin\theta)$ and let $\{\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$ be the right-handed system of unit vectors in the directions of increasing R, θ, and ϕ, respectively. I reserve the use of the term *radial coordinate* to R only and refer to r as *transverse coordinate*.

Let a denote the *centroidal radius*, namely the radius of the circle traced out by the centroid of a typical meridional cross section $\mathcal{D}^c$ of the core as it revolves about the axis of symmetry. Let $\mathcal{R}^i$ denote a spherical ball of radius $3a$ centered on the origin, which I will call the *physical interior*. The left panel of Fig. 1.1 illustrates a typical meridional section, $\mathcal{D}^i$ of $\mathcal{R}^i$. Let $\mathcal{R}^i\backslash\mathcal{R}^c$ denote the compliment of $\mathcal{R}^c$ in $\mathcal{R}^i$ and let $\mathcal{D}^i\backslash\mathcal{D}^c$ denote a typical meridional section thereof. In the left panel of Fig. 1.1 the light blue and dark blue regions ilustrate $\mathcal{D}^c$ and $\mathcal{D}^i\backslash\mathcal{D}^c$ respectively.

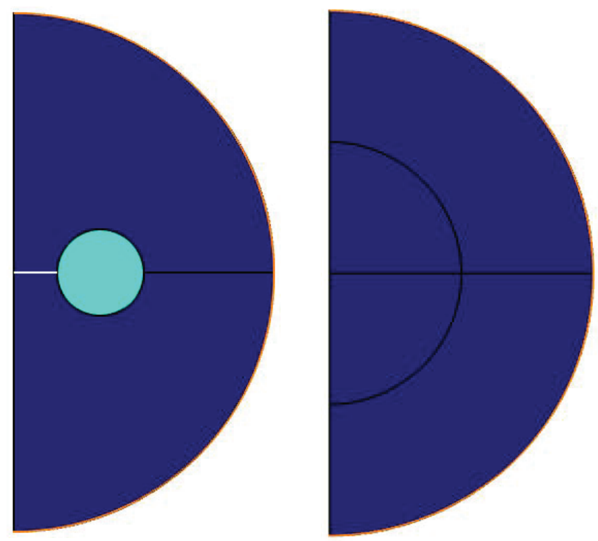

Figure 1.1 Schematic views of meridional sections of $\mathcal{R}^i$ (left panel) and $\mathcal{Q}$ (right panel). In the left panel the light blue region is a section of $\mathcal{R}^c$ and the dark blue region is a section of $\mathcal{R}^i\backslash\mathcal{R}^c$. The short white segment there is a section of the diaphragm, $\mathcal{S}^d$.

To simulate the motion in the (unbounded) physical exterior $\mathcal{R}\backslash\mathcal{R}^i$ I employ a change of position coordinates, namely KELVIN Inversion (Ref. 2), the effect of which is to map $\mathcal{R}\backslash\mathcal{R}^i$ to a bounded *proxy domain*, $\mathcal{Q}$. With this aim in mind let (q, ϑ, φ) be spherical coordinates that cover $\mathcal{Q}$ and are related to the spherical coordinates (R, θ, ϕ) that cover $\mathcal{R}\backslash\mathcal{R}^i$

by the followng transformation rules:

$$\left.\begin{array}{ccccc} q = a^2/R & , & \vartheta = \theta & , & \varphi = \phi \\ q\dfrac{\partial}{\partial q} = -R\dfrac{\partial}{\partial R} & , & \dfrac{\partial}{\partial \vartheta} = \dfrac{\partial}{\partial \theta} & , & \dfrac{\partial}{\partial \varphi} = \dfrac{\partial}{\partial \phi} \end{array}\right\} . \tag{1.1}$$

In the same spirit let (ϖ, φ, ζ) be cylindrical coordinates related to the spherical coordinates (q, ϑ, φ) by $(\zeta, \varpi) = (q\cos\vartheta, q\sin\vartheta)$. Let $\{\hat{\mathbf{u}}_q, \hat{\mathbf{u}}_\vartheta, \hat{\mathbf{u}}_\varphi\}$ be the right-handed system of unit vectors in the directions of increasing q, ϑ, and φ, respectively, and let $\{\hat{\mathbf{u}}_\varpi, \hat{\mathbf{u}}_\varphi, \hat{\mathbf{u}}_\zeta\}$ be the right-handed system of unit vectors in the directions of increasing ϖ, φ, and ζ, respectively. The dark blue region in the right panel of Fig. 1.1 illustrates a meridional section $\mathcal{D}_q$ of $\mathcal{Q}$.

Fig. 1.1 thus illustrates three domains, *i.e.* $\mathcal{D}^c$ and $\mathcal{D}^i\backslash\mathcal{D}^c$ in the left panel and $\mathcal{D}_q$ in the right panel. The present simulation solves three boundary-value problems simultaneously, one in each of these three domains. I will conclude this section with statements of the specific field equations that hold in these three domains.

Consider $\mathcal{R}^c$ first. The present simulation assumes that $\operatorname{curl}\mathbf{u} = Ar\hat{\mathbf{e}}_\phi$ there, in which A is a constant. One may satisfy $\operatorname{div}\mathbf{u} = 0$ by the representation $\mathbf{u} = -\operatorname{curl}(\hat{\mathbf{e}}_\phi\Psi/r)$, in which the scalar Ψ is the Stokes stream function. After some derivation one finds that this representation takes $\operatorname{curl}\mathbf{u} = Ar\hat{\mathbf{e}}_\phi$ to

$$\nabla^{(\mathrm{m})}\bullet[(1/r)\nabla^{(\mathrm{m})}\Psi] = Ar\ , \tag{1.2}$$

in which $\nabla^{(\mathrm{m})}$ is the *meridional gradient* operator in physical domain variables, whose expansion in cylindrical coordinates is $\nabla^{(\mathrm{m})} := \hat{\mathbf{e}}_r\,\partial/\partial r + \hat{\mathbf{k}}\,\partial/\partial z$.

Consider next the domain $\mathcal{R}^i\backslash\mathcal{R}^c$. One may satisfy $\operatorname{curl}\mathbf{u} = \mathbf{0}$ there by the representation $\mathbf{u} = \nabla\Phi = \nabla^{(\mathrm{m})}\Phi$, in which Φ is the velocity potential, provided the domain $\mathcal{R}^i\backslash\mathcal{R}^c$ is simply connected. To this end let $\mathcal{S}^d$ denote a *diaphragm*, namely the figure $z = 0$, $0 \le r \le r_{\mathrm{ie}}$, in which r_{ie} is the value of r at the *inner equator* of the ring. Thus, henceforth, $\mathcal{R}\backslash\mathcal{R}^c$ is bounded internally by both $\mathcal{R}^c$ and the two sides of $\mathcal{S}^d$. A feature of $\mathcal{S}^d$ is that it allows Φ to suffer a discontinuity across it even if, as here, the problem statement allow no such discontinuity of the velocity components.

Let (u_r, u_ϕ, u_z) denote the scalar components of $\mathbf{u}$ relative to the system $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{k}}\}$ and note that the only two nontrivial components of $\mathbf{u} = \nabla^{(\mathrm{m})}\Phi$ are

$$u_r = \partial\Phi/\partial r \quad , \quad u_z = \partial\Phi/\partial r\ . \tag{1.3$_{1,2}$}$$

I assume that u_r is zero on both sides of the diaphragm. From this assumption, $(1.3)_1$, and the assumed axisymmetry of the problem one concludes that the values of Φ on the two sides of the diaphragm are piecewise uniform. Let $\Phi_>$ and $\Phi_<$ denote these piecewise-uniform values of Φ on the top and bottom of the diaphragm, respectively.

Note from $(1.3)_{1,2}$ that $u_r dr + u_z dz$ is an exact differential, namely $d\Phi$. For any contour $\mathcal{C}$ in $\mathcal{D}^i\backslash\mathcal{D}^c$ that starts on the bottom of the diaphragm and ends on the top we have

$$\int_{\mathcal{C}}(u_r dr + u_z dz) = \Phi_> - \Phi_< := C\ , \tag{1.4}$$

in which I will call the constant C *the circulation* about the core.

After some derivation one finds that representation $\mathbf{u} = \nabla^{(\mathrm{m})}\Phi$ takes $\operatorname{div}\mathbf{u} = 0$ to

$$\nabla^{(\mathrm{m})}\bullet(r\nabla^{(\mathrm{m})}\Phi) = 0\ , \tag{1.5}$$

which is the field equation in $\mathcal{R}^i\backslash\mathcal{R}^c$.

Consider, finally, the domain $\mathcal{Q}$. After some derivation one finds that the transformation rules (1.1) take (1.5) to

$$\nabla_q^{(\mathrm{m})}\bullet[\varpi(a^2/q^2)\nabla_q^{(\mathrm{m})}\Phi] = 0\ , \tag{1.6}$$

in which $\nabla_q^{(\mathrm{m})}$ is the the *meridional gradient* operator in proxy domain variables, whose expansion in cylindrical coordinates is $\nabla_q^{(\mathrm{m})} := \hat{\mathbf{u}}_\varpi\,\partial/\partial\varpi + \hat{\mathbf{u}}_\zeta\,\partial/\partial\zeta$ and in which on may take q^2 to be an abbreviation for $\varpi^2 + \zeta^2$.

2. Conditions on the portal

If it were possible to solve the field equation for Φ [namely (1.5)], in both $\mathcal{R}^i\backslash\mathcal{R}^c$ (the region between the core and the the sphere $R = 3a$) and in $\mathcal{R}\backslash\mathcal{R}^i$ (the region exterior to the sphere $R = 3a$) one would impose two continuity conditions at their interface, namely: (a) continuity of the value of Φ; and (b) continuity of the normal component of the flux vector $r\nabla^{(\mathrm{m})}\Phi$ referred to a common unit normal applicable on both sides of the interface. Here Kelvin

Inversion replaces $\mathcal{R}\backslash\mathcal{R}^i$ by its proxy $\mathcal{Q}$. According to (1.1) the image of the spherical boundary $R = 3a$ of $\mathcal{R}^i\backslash\mathcal{R}^c$ is the spherical boundary $q = a/3$, of $\mathcal{Q}$. Since these two spherical boundaries have different sizes I refer to each of them as a *portal* to the other and to the counterpart of the two continuity conditions as *portal conditions*.

The first portal condition is a Dirichlet condition for the boundary-value problem whose field equation is (1.6) and it requires that the value of Φ at a typical point on the sphere $q = a/3$ equals the value of Φ at the corresponding point on the other side of the portal.

The second portal condition is a Flux/Source boundary condition for the boundary-value problem whose field equation is (1.5) and it requires that the expression $-\hat{\mathbf{n}}\bullet(r\nabla^{(\mathrm{m})}\Phi)$ (in which $\hat{\mathbf{n}}$ is the unit normal at a typical point on the sphere $R = 3a$ directed out of $\mathcal{R}^i\backslash\mathcal{R}^c$) equals the value it takes after transformation to proxy-domain variables by (1.1) and evaluation at the corresponding point on the other side of the portal. After some derivation (which space limitations prevent me from including here) this Flux/Source boundary condition becomes

$$-\hat{\mathbf{n}}\bullet(r\nabla^{(\mathrm{m})}\Phi) = \frac{\varpi}{q}\left(\varpi\frac{\partial\Phi}{\partial\varpi} + \zeta\frac{\partial\Phi}{\partial\zeta}\right) . \qquad (2.1)$$

3. Conditions on the core boundary

The conditions on the core boundary assert that each side of it is impermeable.

In the boundary-value problem whose field equation is (1.5) the impermeability condition is a Flux/Source boundary condition, namely a specification of the value of $-\hat{\mathbf{n}}\bullet(r\nabla^{(\mathrm{m})}\Phi)$ there. Now $-W\hat{\mathbf{k}}$ is the velocity of the ring relative to the remote undisturbed fluid so the impermeabililty condition requires that $\mathbf{u}\bullet\hat{\mathbf{n}} = (-W\hat{\mathbf{k}})\bullet\hat{\mathbf{n}} = -Wn_z$. In the mean time $\mathbf{u} = \nabla^{(\mathrm{m})}\Phi$ so the Flux/Source boundary condition becomes

$$-\hat{\mathbf{n}}\bullet(r\nabla^{(\mathrm{m})}\Phi) = rWn_z \; . \qquad (3.1)$$

In regard to the boundary value problem whose field equation is (1.2) note that the nontrivial components of $\mathbf{u} = -\mathrm{curl}\,(\hat{\mathbf{e}}_\phi\Psi/r)$ are

$$u_r = (1/r)\partial\Psi/\partial z \quad , \quad u_z = -(1/r)\partial\Psi/\partial r \; , \qquad (3.2)_{1,2}$$

from which we have

$$u_r = \frac{1}{r}\frac{\partial}{\partial z}\big(\Psi - \frac{Wr^2}{2}\big) \; , \; u_z + W = -\frac{1}{r}\frac{\partial}{\partial r}\big(\Psi - \frac{Wr^2}{2}\big) \; . \qquad (3.3)_{1,2}$$

The components $(u_r, u_z + W) := (v_r, v_z)$ are those of the fluid velocity $\mathbf{v}$ relative to an observer riding with the ring and the expression $\Psi - Wr^2/2$ denotes the corresponding relative stream function, which must be constant on an impermeable surface that appears stationary to such an observer (as here). Thus $\Psi = Wr^2/2 +$ constant on the core boundary. To determine the constant note that the $\mathbf{u}$ in $\mathcal{R}^i\backslash\mathcal{R}^c$ is solenoidal so Ψ has meaning there. If one arranges $(3.2)_2$ in the form $\partial\Psi/\partial r = -r\,u_z$ integrates with respect to r from $r = 0$ to $r = r_{\mathrm{ie}}$ and applies the boundary condition $\Psi = 0$ on $r = 0$ one deduces a formula for Ψ at $r = r_{\mathrm{ie}}$ and, thence, at the Dirichlet condition

$$\Psi = (1/2)W(r^2 - r_{\mathrm{ie}}^2) - \int_0^{r_{\mathrm{ie}}} ru_z dr \; . \qquad (3.4)$$

4. Conditions for Φ on $z = 0$ and $\zeta = 0$

In all the computations reported herein the velocity potential Φ was taken to be an odd function of z in $\mathcal{R}^i\backslash\mathcal{R}^c$ and of ζ in $\mathcal{Q}$. Specifically, Φ is subject to the Dirichlet condition $\Phi = 0$ on the annular part of the equitorial plane $z = 0$ between the outer equator of the ring and the outer boundary of $\mathcal{R}^i$. Furthermore Φ is subject to the Dirichlet conditions $\Phi = C/2$ and $\Phi = -C/2$ on the upper and lower sides of the diaphragm, respectively. Similarly, Φ is subject to the Dirichlet condition $\Phi = 0$ on the whole equitorial disk $\zeta = 0$ in the proxy domain .

5. On the coefficient A in $\omega_\phi = Ar$

Note that the only nontrivial component of $\mathrm{curl}\,\mathbf{u} = Ar\hat{\mathbf{e}}_\phi$ is $\partial u_r/\partial z - \partial u_z/\partial r = Ar$. If one integrates with respect to area over the core cross section $\mathcal{D}^c$ and rewrites the left member by means of Green's theorem in the plane one gets

$$\int_{\partial\mathcal{D}^c}(u_r dr + u_z dz) = \iint_{\mathcal{D}^c} Ar\,drdz \; . \qquad (5.1)$$

The integral in the left member is similar to the one in the left member of (1.4), where I denoted its value by C. The difference between these two integrals concerns their integration paths: specifically, the former lies in $\mathcal{D}^i\backslash\mathcal{D}^c$—though it may be shrunk down to hug the core boundary—while the latter lies on the inside of that boundary. One aim of the present simulation is to elminate slip between the inside and the outside of the core boundary. To satisfy this aim the integral in the left member of (5.1) must also equal C. Now A is constant so the right member of (5.1) is equivalent to $A(\iint_{\mathcal{D}^c} r\,drdz/\iint_{\mathcal{D}^c} drdz)\iint_{\mathcal{D}^c} drdz$, in which the quotient in parentheses is just the centroidal radius, a. If one defines the length δ such that $\iint_{\mathcal{D}^c} drdz := \pi\delta^2$, equation (5.1) reduces to the form $C = Aa\pi\delta^2$, so

$$A = C/(a\pi\delta^2)\ . \tag{5.2}$$

6. Arc-length variable over the a meridional section of the core boundary

The unit normal vector $\hat{\mathbf{n}}$ on the core boundary directed out of $\mathcal{D}^i\backslash\mathcal{D}^c$ points into the core and lies in a typical meridional cross section. Then the unit vector $\hat{\mathbf{t}}$ defined by $\hat{\mathbf{t}} = \hat{\mathbf{e}}_\phi \times \hat{\mathbf{n}}$ lies in the same meridional cross section, is also tangent to the core boundary, and points in the counter-clockwise sense in the left panel of Fig. 1.1. The operator $\hat{\mathbf{t}}\bullet\nabla^{(\mathrm{m})} = (\hat{\mathbf{e}}_\phi\times\hat{\mathbf{n}})\bullet\nabla^{(\mathrm{m})} = n_z\partial/\partial r - n_r\partial/\partial z$ thus represents differentiation with respect to arc length along the core boundary in the direction of $\hat{\mathbf{t}}$. Let

$$ds_{\mathrm{my}}/ds := n_z\,\partial s_{\mathrm{my}}/\partial r - n_r\,\partial s_{\mathrm{my}}/\partial z\ , \tag{6.1}$$

in which s_{my} is a dependent variable to be computed as the solution of a Boundary Partial Differential Equation. In COMSOL's Weak Form Boundary PDE Physics Interface the software has an input field labled Weak Expression where I entered the equivalent of

$$(ds_{\mathrm{my}}/ds - 1)\mathrm{test}(ds_{\mathrm{my}}/ds)\ . \tag{6.2}$$

I also specified the Dirichlet condition $s_{\mathrm{my}} = 0$ on the outer equator of the ring and accepted the default Null Flux condition at extremities, namely the top and bottom of the inner equator.

7. Circulation and propagation speed as solutions of a matrix equation

The present calculations employ a normalization that requires some motivation. Lighthill (1985) (Ref. 3) derives a the large-R asymptotic expansion of the velocity potential in a class of problems that includes the present one. The appropriate special case of Lighthill's expansion [equation (290), *Ibid.*] is

$$\Phi = \Phi_\infty + \mathbf{G}\bullet\nabla[1/(4\pi R)] + O(R^{-3})\ , \tag{7.1}$$

in which $\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2$ and

$$\mathbf{G}_1 := \iint\limits_{\mathcal{S}} \Phi\hat{\mathbf{n}}\,dA \quad , \quad \mathbf{G}_2 := -\iint\limits_{\mathcal{S}} \mathbf{R}(\mathbf{u}\bullet\hat{\mathbf{n}})dA \qquad (7.2)_{1,2}$$

[equation (296), *Ibid.*]. Here $\mathcal{S}$ is a closed surface whose interior and exterior of are both filled with fluid in solenoidal motion while the motion in the exterior is also irrotational, the unit normal $\hat{\mathbf{n}}$ is directed out of the exterior of $\mathcal{S}$, and $\mathbf{R}$ is the local position vector on $\mathcal{S}$. One may deform $\mathcal{S}$ without altering the values of $\mathbf{G}_1$ and $\mathbf{G}_2$ provided one abides by the aformentioned conditions. One may, in particular, deform $\mathcal{S}$ until it coincides with $\mathcal{S}^d\cup\Sigma$, in which $\mathcal{S}^d$ is the diaphragm and Σ is the exterior side of the core boundary.

In the present axisymmetric problem each of the vectors $\mathbf{G}_1$ and $\mathbf{G}_2$ has only one nontrivial vector component namely the one proportional to $\hat{\mathbf{k}}$. If the ring propagates in the negative z-direction with velocity $-W\hat{\mathbf{k}}$, in which $W > 0$ one may show that $\mathbf{G} = -G\hat{\mathbf{k}}$, in which $G > 0$.

Lighthill shows that $\mathbf{G}_2$ equals the product of the volume enclosed by $\mathcal{S}$ with the vector velocity of its centroid [equation (301) *Ibid.*]. This rule leads to the result $\mathbf{G}_2 = (-W\hat{\mathbf{k}})(2\pi a)(\pi\delta^2)$. The contribution to $\mathbf{G}_1$ from $\mathcal{S}^d$ proves to be $(-\hat{\mathbf{k}})\pi r_{\mathrm{ie}}^2 C$, while the full expansion of $(-\hat{\mathbf{k}})\bullet\mathbf{G}$ proves to be

$$G = \pi r_{\mathrm{ie}}^2 C - \iint_\Sigma \Phi n_z dA + W(2\pi a)(\pi\delta^2)\ . \tag{7.3}$$

Now G, called *the total dipole strength*, has the dimesions of length^4/time. In the present calculations I apply the normalization $G = 1\,\mathrm{m}^4/\mathrm{s}$.

Let v_t denote the component of fluid velocity tangent to the core boundary as perceived by an observer who propagates with the ring. Let $(v_t)_{\mathrm{ex}}$ and $(v_t)_{\mathrm{in}}$ denote the values of v_t on the interior and exterior sides of the core boundary, respectively, and let $\Delta v_t := (v_t)_{\mathrm{ex}} - (v_t)_{\mathrm{in}}$. Let the sign conventions for $(v_t)_{\rm ex}$ and $(v_t)_{\rm in}$ be the same, namely clockwise positive in the left panel of Fig. 1.1.

The boundary-value problems for Ψ and Φ developed thus far result in solutions that depend linearly on each of the parameters W and on C and a similar statement applies G and Δv_t. We thus have identities of the form

$$G_C C + G_W W = G \tag{7.4}$$

$$(\Delta v_t)_C C + (\Delta v_t)_W W = \Delta v_t \tag{7.5}$$

in which an expression with the subscript $(\,.\,)_C$ denotes evaluation in the case $(C, W) = (1, 0)$ and an expression with the subscript $(\,.\,)_W$ denotes evaluation in the case $(C, W) = (0, 1)$.

In all the calculations reported herin I took $\delta = a/2$. Let C_0 denote the solution of (7.4) corresponding to the case $W = 0$, $G = 1\,\mathrm{m}^4/\mathrm{s}$. Evaluation of the field variables corresponding to the case $(C, W) = (C_0, 0)$ then generates a solution one may characterize as *pure circulation without propagation.* The blue curve in Fig. 7.1 illustrates the distribution of Δv_t in this case.

Alternatively, let W_0 denote the solution of (9.2) corresponding to the case $C = 0$, $G = 1\,\mathrm{m}^4/\mathrm{s}$. Evaluation of the field variables corresponding to the case $(C, W) = (1, W_0)$ then generates a solution one may characterize as *pure propagation without circulation.* The green curve in Fig. 7.1 illustrates the distribution of Δv_t in this case.

In both of these examples the slip is largest at the inner equator, where $r = r_{\rm ie}$. Consider a third example, in which the terms in (7.5) are all evaluated at the most problematic point, namely $r = r_{\rm ie}$, and the right member set equal to zero there. If, as before $G = 1\,\mathrm{m}^4/\mathrm{s}$ then one may arrange the system (7.4), (7.5) as a matrix equation, namely

$$\begin{bmatrix} G_C & G_W \\ (\Delta v_t)_{C\,\rm ie} & (\Delta v_t)_{W\,\rm ie} \end{bmatrix} \begin{pmatrix} C \\ W \end{pmatrix} = \begin{pmatrix} 1\,\mathrm{m}^4/\mathrm{s} \\ 0 \end{pmatrix} \;, \tag{7.6}$$

Let $(C_1\; W_1)^T$ be the solution of (7.6) for $(C\; W)^T$ as defined by CRAMER's rule. Evaluation of the field variables corresponding to this case then generates a solution that one may chracterize as a *circulating and propagating solution with no-slip at one point.* The red curve in Fig. 7.1 illustrates the distribution of Δv_t in this case.

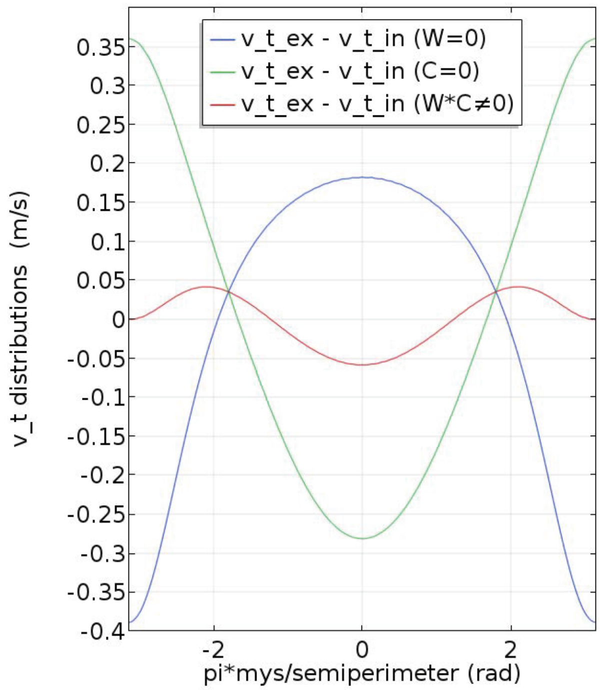

Figure 7.1 Distributions of the slip velocity, $(v_t)_{\rm ex} - (v_t)_{\rm in}$ across the core boundary. Here W denotes the downward propagation velocity of the ring relative to the remote undisturbed fluid and C denotes the circulation about the core. Results shown are for a core of circular cross section, which exhibits unwanted slip across the core boundary.

8. The no-slip condition as an optimization problem

A model whose results exhibit nonzero slip across the core boundary, as shown in Fig 7.1 is physically unrealistic, since it necessitates a discontinuity of pressure across that boundary. The present model addresses this defect by employing a general family of noncircular cross sections

To this end let $\mathfrak{D}_0^c$ denote the result of a Geometry Sequence that generates a core boundary in the form of a circle of radius $\delta = a/2$ centered on the point $(R, Z) = (a, 0)$. Now let $\partial\mathfrak{D}_1^c$ satisfy the parametric equations

$$R_{\partial 1} = a + P_\partial \cos\alpha \quad , \quad Z_{\partial 1} = P_\partial \sin\alpha \;, \tag{8.1$_{1,2}$}$$

in which $P_\partial = \delta\big[1 + \sum_1^N \epsilon_n \cos(n\alpha)\big]$ and in which the coefficients ϵ_n, $n \in \{1, \ldots, N\}$ are shape parameters that will be control variables in an optimization problem. Here α has the values $-\pi$, 0, and π

at the bottom of the diaphragm, the outer equator, and the top of the diaphragm, respectively. Note that α increases in the same direction as does the arc-length variable, $s_{\rm my}$ of §**6**.

Now the transverse coordinate of the centroid of $\mathcal{D}_1^c$ (which I denote by R_{1c}) differs from that of $\mathcal{D}_0^c$ (namely a). Likewise, the cross sectional area of $\mathcal{D}_1^c$ [which I denote by $\pi(M\delta)^2$] differs from that of $\mathcal{D}_0^c$ (namely $\pi\delta^2$). One may interpret M as a *magnification factor*. To undo the first of these changes, construct the transformation $\mathcal{D}_1^c \mapsto \mathcal{D}_2^c$ as a simple transverse displacement that ensures that the transverse coordinate of the centroid of $\mathcal{D}_2^c$ *is* the same as that of $\mathcal{D}_0^c$ (namely a). To undo the second of the aforementioned changes, construct the transformation $\mathcal{D}_2^c \mapsto \mathcal{D}_3^c$ as a rescaling by the factor M^{-1} about the centroid of $\mathcal{D}_2^c$. Such a rescaling ensures that the cross sectional area of $\mathcal{D}_3^c$ is the same as that of $\mathcal{D}_0^c$ (namely $\pi\delta^2$) and that the value of the transverse coordinate r corresponding to the centroid of $\mathcal{D}_3^c$ agrees with the value of the transverse coordinate R corresponding to the centroid of $\mathcal{D}_2^c$ (namely a).

I employed COMSOL's Moving Mesh Interface. At the domain level (over all of $\mathcal{D}^i$) I chose the option Free Deformation. On the axis of symmetry, and on the parts of the equitorial plane exterior to the core I chose Zero Normal Mesh Displacement. On the portal I chose Prescribed Mesh Displacement equal to zero, and on the core boundary I chose Prescribed Mesh Displacement equal to

$$d_r = (M^{-1}P_\partial - \delta)\cos\alpha + M^{-1}(a - R_{1c}) \ , \quad (8.2)_1$$
$$d_z = (M^{-1}P_\partial - \delta)\sin\alpha \ . \quad (8.2)_2$$

In order enable COMSOL to read the right members of $(8.2)_{1,2}$ at a typical point (R, Z) on $\partial\mathcal{D}^c$, I introduced definitions of Variables equivalent to $\cos\alpha := (R-a)/\delta$ and $\sin\alpha := Z/\delta$. The definition of P_∂ involves $\cos(n\alpha)$, $n \in \{1, \ldots, N\}$. To express $\cos(n\alpha)$ in terms of $\cos\alpha$ (and thus R) note that $\cos(n\alpha) = T_n(\cos\alpha)$, in which $x \mapsto T_n(x)$ is the Chebyshev polynomial of the first kind of order n. Here $T_0(x) := 1$, $T_1(x) := x$, while the $T_n(x)$ for all $n \in \{2, 3, \ldots\}$ are determined by the recursion relation $T_{n+2}(x) := 2xT_n(x) - T_{n-1}(x)$. To evaluate M and R_{1c} note that

$$\left.\begin{aligned} \pi(M\delta)^2 &:= \textstyle\iint_{\mathcal{D}^c} dA \\ R_{1c} &:= \textstyle\iint_{\mathcal{D}^c} R\,dA \big/ \iint_{\mathcal{D}^c} dA \end{aligned}\right\} . \quad (8.3)$$

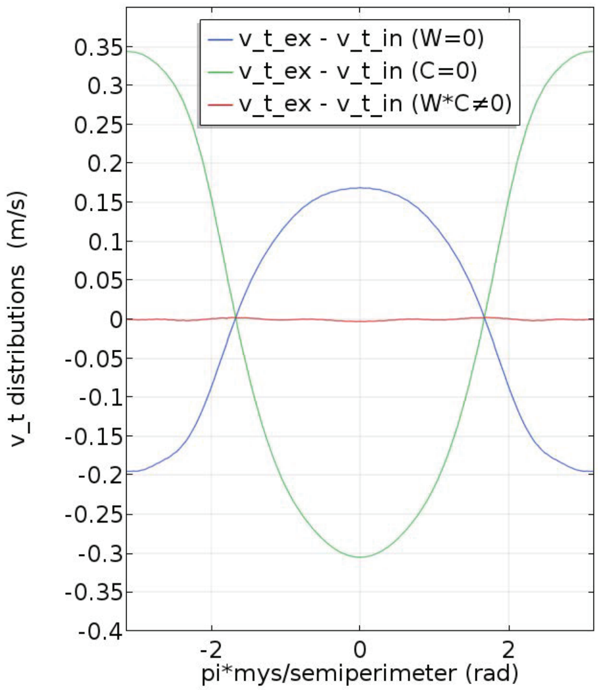

Figure 8.1 Legend same as that of Fig. 7.1 except that this time, the results shown are for a noncircular core shape after optimization to minimize slip across the core boundary in the case of a propagating ring with circulation.

One may reduce the double integrals in the right member to single integrals with respect to α, namely

$$\left.\begin{aligned} \textstyle\iint_{\mathcal{D}^c} dA &= -\textstyle\int_{-\pi}^{\pi} Z\,dR/d\alpha\,d\alpha \\ \textstyle\iint_{\mathcal{D}^c} R\,dA &= -\textstyle\int_{-\pi}^{\pi} RZ\,dR/d\alpha\,d\alpha \end{aligned}\right\} . \quad (8.4)$$

The present model employed COMSOL's `integrate` operator to evaluate the integrals with respect to α in (8.4).

To determine the shape parameters ϵ_n, $n \in \{1, \ldots, N\}$ with COMSOL's Optimization tools I specified an Objective Function equivalent to

$$F = C_1^{-2} \int_{\partial\mathcal{D}^c} (\Delta v_t)^2 ds \cdot \int_{\partial\mathcal{D}^c} ds \quad (8.5)$$

and endeavored to minimize F. I found it useful to manually cycle through three root models, of which one utilized a Parametric Sweep Study Step, another utlilized a Sensitivity Physics Interface with an accompanying Study Step, and the third utilized an Optimization Study Step (with the NELDER-MEAD option). I found that with $N = 12$ I could reduce F by three orders of magnitude. Fig. 8.1 illustrates the results of this optimization. Though the red line exhibits nonzero residual slip across the core of a propagating ring with circulation that residual is a considerable improvement over the data shown in Fig. 7.1.

9. Computation of Ψ in $\mathcal{R}^i \backslash \mathcal{R}^c$

Having u_r and u_z in $\mathcal{D}^i \backslash \mathcal{D}^c$ as derivatives of Φ I computed the corresponding STOKES stream function Ψ by specifying

$$\left(\frac{\partial \Psi}{\partial r} + r u_z\right) \mathrm{test}\left(\frac{\partial \Psi}{\partial r}\right) + \left(\frac{\partial \Psi}{\partial z} - r u_r\right) \mathrm{test}\left(\frac{\partial \Psi}{\partial z}\right) \tag{9.1}$$

in the input field for a Weak Form PDE Physics interface. Note that the first factor in each of the two products in the above sum is zero by $(3.2)_{1,2}$. I applied continuity boundary conditions on both of the equatorial segments outside the core, applied the DIRICLET boundary condition $\Psi = 0$ on the axis of symmetry, and accepted the default Null Flux condition on the core boundary and the portal. The result, as shown in Fig. 9.1, is the distribution of Ψ relative to an observer at rest relative to the remote fluid.

If, in the post-processing stage, one replaces Ψ by $\Psi - Wr^2/2$ as in $(3.2)_{1,2}$ the result is the STOKES stream function relative to an observer who propagates with the ring. Fig. 9.2 shows the results.

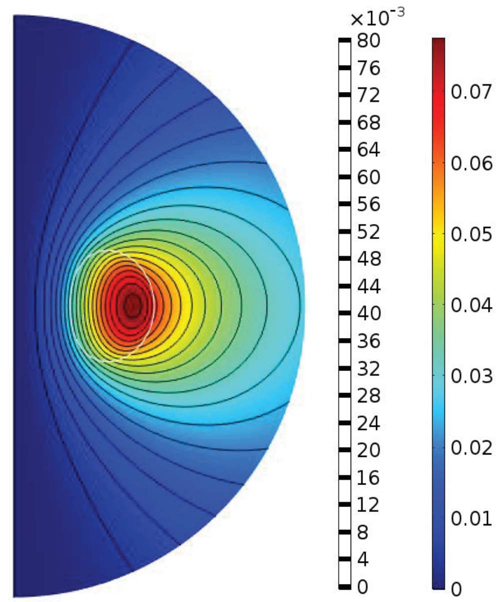

Figure 9.1 Ψ in $\mathcal{D}^i$ relative as seen by an observer at rest relative to the remote undisturbed fluid.. The white contour is the core boundary. The increment of Ψ between contours is $0.04 \times 10^{-3}\, G/a^2$.

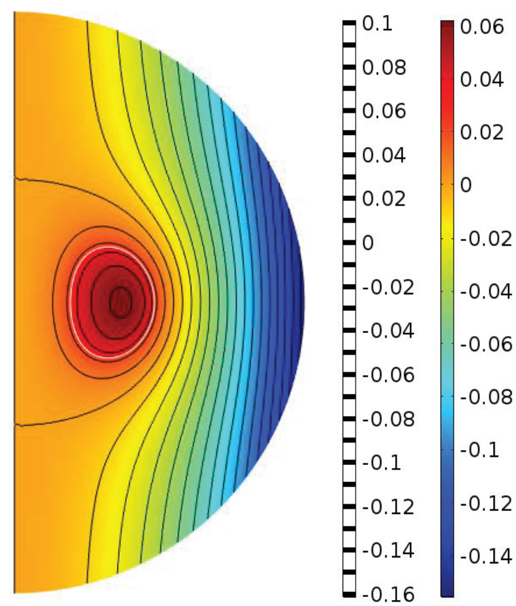

Figure 9.2 Legend similar to that of Fig. 9.1, except that now the results are as seen by an observer propagating with the ring.

10. References

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