Application of an Electromagnetic Analogy in the Simulation of a Problem in Classical Hydrodynamics

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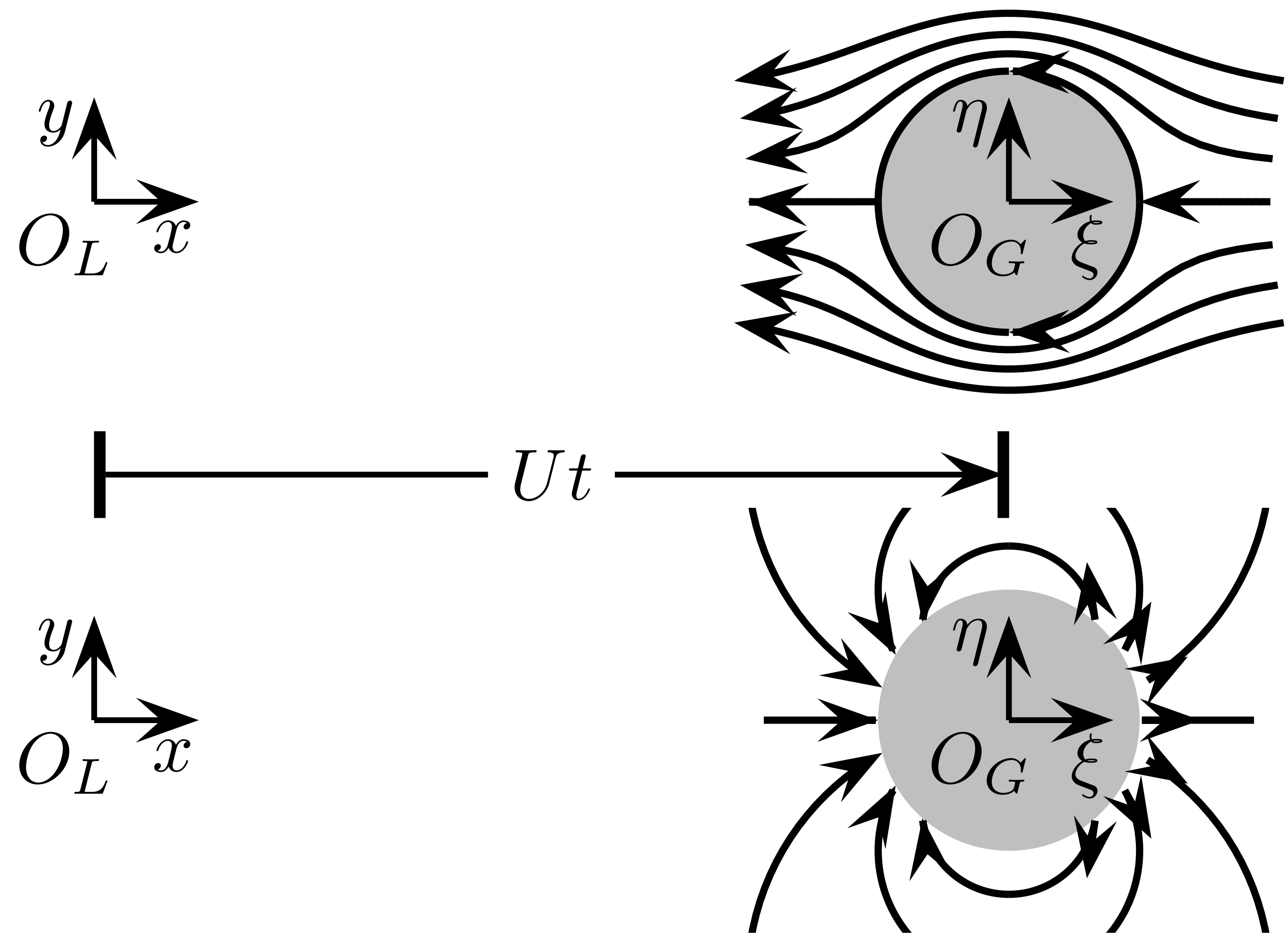
Objective

Use COMSOL to simulate a traveling vortex structure in an unbounded domain in a case where an analytic solution* is available

* Lamb, Sir Horace *Hydrodynamics.* Sixth edition, Cambridge University Press 1932, p 245

Streamlines relative to frames $\mathcal{F}_G$ (upper) and $\mathcal{F}_L$ (lower)



Assumptions

Motion is two-dimensional: velocity vector ($\mathbf{u}_L$ or $\mathbf{u}_G$) everywhere parallel to a plane $\mathcal{P}$;

Fluid is inviscid, incompressible and of uniform density, ρ;

Fluid at rest relative to $\mathcal{F}_L$ at large distances from moving structure

Motion stationary relative to $\mathcal{F}_G$.

Partition into subdomains

$\mathcal{D} = \mathcal{D}_b \cup \mathcal{D}_e$, in which:

$\mathcal{D}_b$ is a bounded (simply connected) subdomain that contains the vortex structure;

$\mathcal{D}_e$ is the unbounded (doubly connected) exterior of $\mathcal{D}_b$ (in which the motion is irrotational).

LAMB'S ANALYTIC SOLUTION IN $\mathcal{D}_e$, I

We have

$$\underbrace{\partial u_\xi/\partial\xi + \partial u_\eta/\partial\eta = 0}_{\text{incompressibility}}\,,\ \underbrace{\partial u_\eta/\partial\xi - \partial u_\xi/\partial\eta = 0}_{\text{irrotationality}}\,.$$

The representation

$$u_\xi = -\partial\psi_G/\partial\eta \quad , \quad u_\eta = \partial\psi_G/\partial\xi \qquad (A)_{1,2}$$

satisfies incompressibility provided $(\xi,\eta) \mapsto \psi_G$ is single valued and twice differentiable in $\mathcal{D}_e$. Note that $(A)_{1,2}$ takes irrotationality to LAPLACE's equation, $\partial^2\psi_G/\partial\xi^2 + \partial^2\psi_G/\partial\eta^2 = 0$.

LAMB'S ANALYTIC SOLUTION IN $\mathcal{D}_e$, II

Example 1. If U is a positive constant then $\psi_G = U\eta$ is a solution of LAPLACE's equation that corresponds to a uniform stream of speed U in the negative ξ direction

Example 2. If one defines the polar coordinates (ϖ, ϑ) by $\xi = \varpi\cos\vartheta$, $\eta = \varpi\sin\vartheta$ then one can arrange LAPLACE's equation for ψ_G in the form

$$\varpi(\partial/\partial\varpi)[\varpi(\partial\psi_G/\partial\varpi)] + \partial^2\psi_G/\partial\vartheta^2 = 0\,,$$

of which $\psi_L = U(a^2/\varpi)\sin\vartheta$ is a solution, namely the solution relative to $\mathcal{F}_L$ (a is a constant length).

Lamb's analytic solution in $\mathcal{D}_e$, III

Example 3. The superposition of the solutions in *Examples* 1 and 2, namely

$$\psi_G = U(\varpi - a^2/\varpi)\sin\vartheta \ ,$$

represents the irrotational flow past a circular cylinder and is the solution relative to $\mathcal{F}_G$.

LAMB'S ANALYTIC SOLUTION IN $\mathcal{D}_b$, I

The assumption that the motion be stationary relative to $\mathcal{F}_G$ leads to the requirement that the vorticity $\omega_\zeta := \partial u_\eta/\partial\xi - \partial u_\xi/\partial\eta = \partial^2\psi_G/\partial\xi^2 + \partial^2\psi_G/\partial\eta^2$ be constant along every streamline. But the streamlines are contours of constant ψ_G so there must be a function f such that $\partial^2\psi_G/\partial\xi^2 + \partial^2\psi_G/\partial\eta^2 = f(\psi_G)$. LAMB considered the example $f(\psi_G) = -k^2\psi_G$ (for constant k), which leads to the HELMHOLTZ equation

$$\partial^2\psi_G/\partial\xi^2 + \partial^2\psi_G/\partial\eta^2 + k^2\psi_G = 0\,.$$

LAMB'S ANALYTIC SOLUTION IN $\mathcal{D}_b$, II

If one multiplies the-polar coordinate form of the HELMHOLTZ equation by ϖ^2 and defines $k\varpi = w$ one gets

$$w(\partial/\partial w)[(w(\partial\psi_G/\partial w)] + \partial^2\psi_G/\partial\vartheta^2 + w^2\psi_G = 0\ ,$$

of which a trial solution of the form $\psi_G \propto W(w)\sin(\vartheta)$ is possible provided

$$w(d/dw)[w(dW/dw)] + (w^2 - 1)W = \ 0, \qquad (A)$$

which is the BESSEL equation of order 1. The BESSEL function of the first kind, $J_1(w)$ satisfies (A) and is regular at $w = 0$.

LAMB'S ANALYTIC SOLUTION IN $\mathcal{D}_b$, III

The solution $CJ_1(k\varpi)\sin(\vartheta)$ for ψ_G in $\mathcal{D}_b$ (C is a constant) can be continuous with the solution $U(\varpi - a^2/\varpi)\sin\vartheta)$ for ψ_G in $\mathcal{D}_e$ only if they agree at the interface, $\varpi = a$. But the solution in $\mathcal{D}_e$ vanishes for all ϑ on that interface so the solution in $\mathcal{D}_b$ must as well. Therefore $J_1(ka) = 0$, the lowest nontrivial root of which is $ka = 3.83171$. The ϖ-derivatives of these solutions must also agree on the interface. By appeal to BESSEL function identities LAMB showed that $C = 2U/[kJ_0(ka)]$.

Electromagnetic analogy, I

Lamb's equations in vector form, *i.e.*

$$\underbrace{\boldsymbol{\omega}_L := \mathrm{curl}\,_L(\mathbf{u}_L)}_{\text{definition of vorticity}} \quad , \quad \underbrace{\mathbf{u}_L = \mathrm{curl}\,_L(-\psi_L\hat{\mathbf{k}})}_{\text{from incompressibility}} \quad ,$$

resemble those of Ampère's law, *i.e.*

$$\mathbf{J} = \mathrm{curl}\,\mathbf{H} \quad , \quad \mathbf{B} = \mathrm{curl}\,\mathbf{A} \ ,$$

especially when $\mathbf{A} = A_z\hat{\mathbf{k}}$ (as in 2D) and when one can apply the constitutive equation

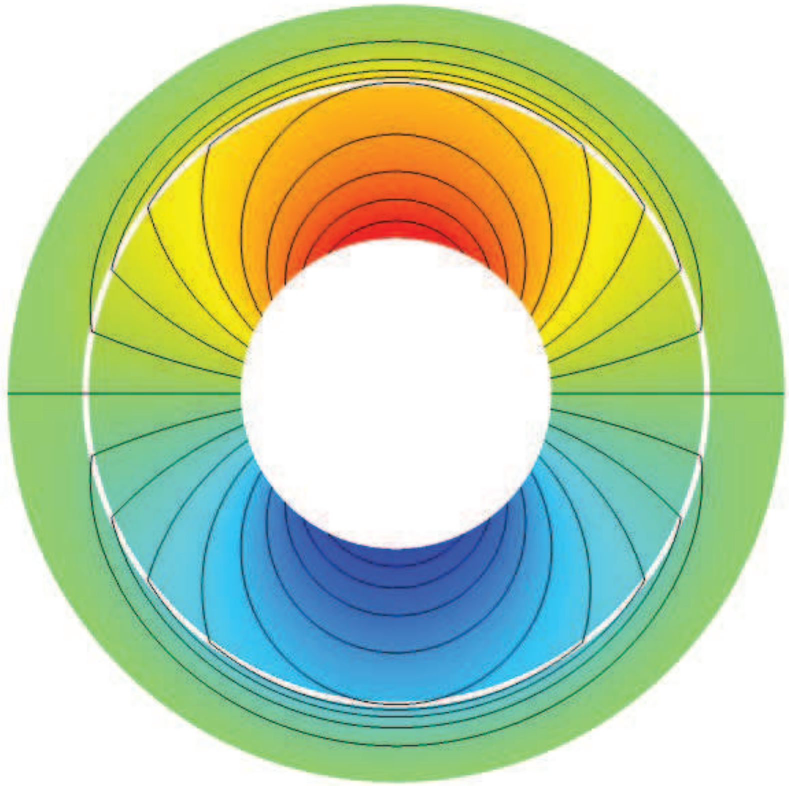
$$\mu_0\mu_r\mathbf{H} = \mathbf{B} \ .$$

Electromagnetic analogy, II

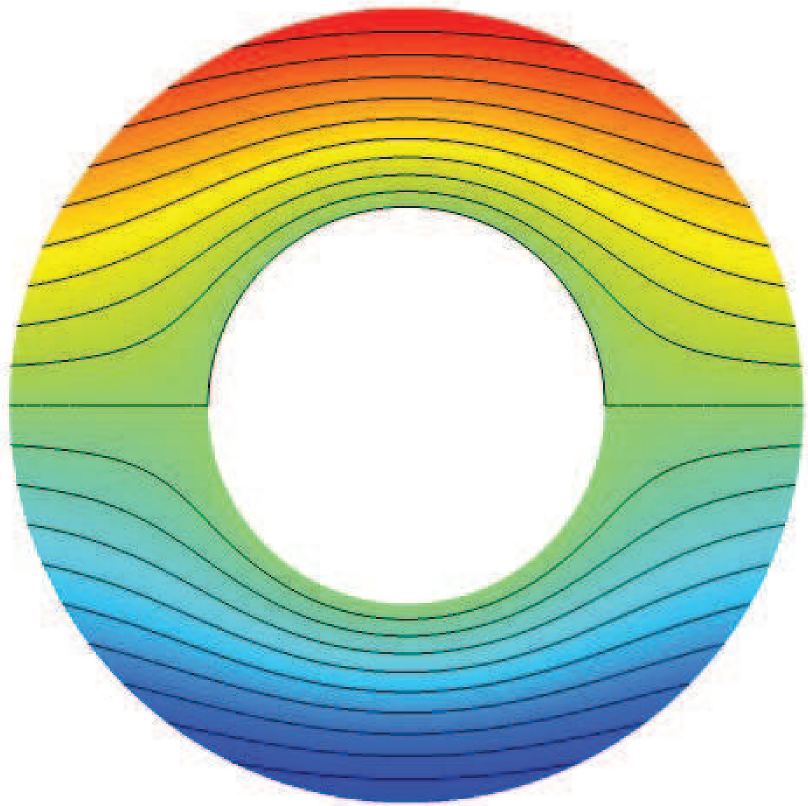
The foregoing hydrodynamic and electromagnetic equations are equivalent under the transformation rule

$$\boldsymbol{\omega}_L = \frac{U}{B_s}\mu_0\mu_r\mathbf{J}\ ,\ \ \mathbf{u}_L = \frac{U}{B_s}\mathbf{B}\ ,\ \ -\psi_L = \frac{U}{B_s}A_z\ ,$$

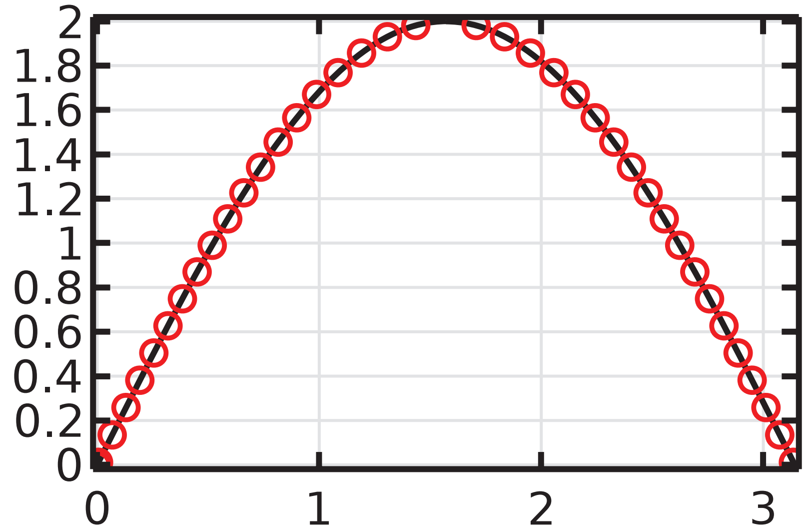
in which B_s is a constant scale for the magnetic flux density. For irrotational motion $\boldsymbol{\omega}_G = \mathbf{0}$, which corresponds to the case, $\mathbf{J} = \mathbf{0}$. Similarly, the boundary conditions for ψ_L transform to boundary conditions for A_z.



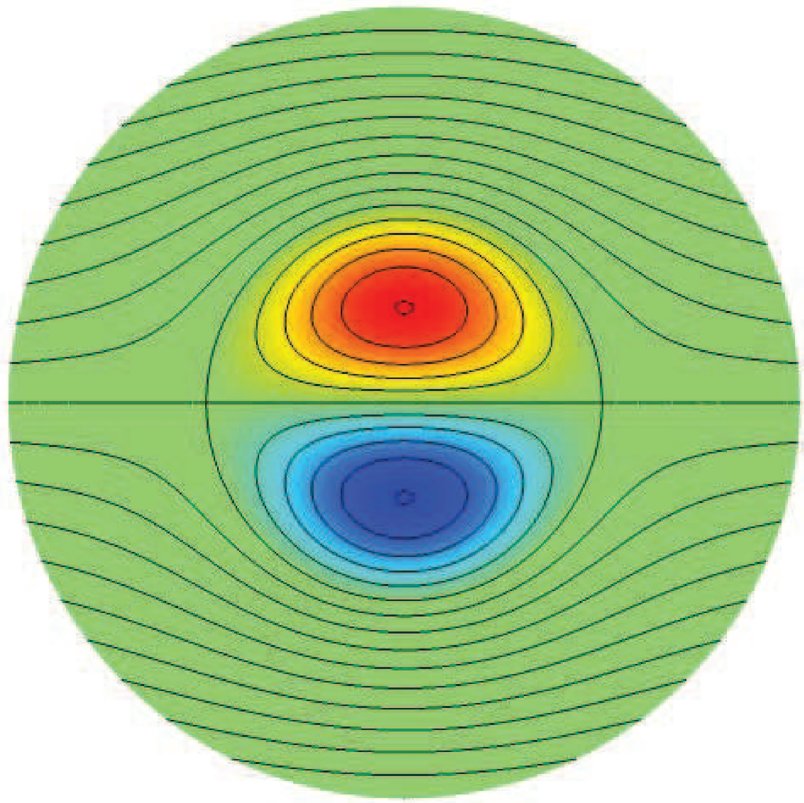
Magnetic potential A_z as seen in $\mathcal{F}_L$.



Stream function $\psi_G = \psi_L + U\eta$ as seen in $\mathcal{F}_G$.



Nondimensional tangential velocity $\mathbf{u}_G \bullet \hat{\mathbf{t}}/U$ versus polar angle ϑ on the inside (red circles) and outside (black line) of (upper half of) $\mathcal{D}_e \cap \mathcal{D}_b$ as seen in $\mathcal{F}_G$.



Vorticity ω_ζ (color) and stream function ψ_G. COMSOL's result for the eigenvalue ka is 3.8317 (contours) as seen in $\mathcal{F}_G$